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On Cesàro means, Kaplan classes and a conjecture of S.P. Robinson ☆

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ABSTRACT

We establish special cases of a conjecture of S.P. Robinson [S.P. Robinson, Approximate identities for certain dual classes, DPhil thesis, University of York, UK, 1996] concerning Cesàro means of certain classes of analytic functions in the unit disk. This has applications, for instance, to the so-called Kaplan classes and subordination under 'linearly accessible' functions.

1. Introduction

1.1. The S.P. Robinson conjecture

The Cesàro polynomial $C_n^{(\alpha)}(z)$ of order $\alpha > 0$ and degree $n \in \mathbb{N}$ is defined as

$$C_n^{(\alpha)}(z) := \sum_{k=0}^{n} \frac{(\alpha+1)_{n-k}}{(n-k)!} \frac{n!}{(\alpha+1)_n} z^k,$$

where $(\beta)_k := \beta(\beta+1)\cdots(\beta+k-1)$ denotes the Pochhammer symbol. For a function $f := \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(\mathbb{D})$, the class of analytic functions in the unit disk $\mathbb{D}$, we call

$$C_n^{(\alpha)}(z, f) := \sum_{k=0}^{n} a_k \frac{(\alpha+1)_{n-k}}{(n-k)!} \frac{n!}{(\alpha+1)_n} z^k = \left(C_n^{(\alpha)} * f\right)(z),$$

the n-th Cesàro mean of order α of f. Here the operator $*$ denotes the Hadamard product. Cesàro means play an important role in approximation theory, summability and Fourier analysis, to name only a few areas. But also in geometric function theory they have found many applications, and their geometric properties have been studied on several occasions (for instance [9,11] and further references therein). One of the striking properties of the sequence $C_n^{(\alpha)}(z, f)$ with $f \in \mathcal{H}(\mathbb{D})$ and α both fixed is that they form an *approximate identity* for $n \to \infty$, converging to f in the sense of compact convergence

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in $\mathbb{D}$. And, frequently, the Cesàro means of f share certain geometric properties with f. In this paper we are dealing with such situations, in particular in the connection with the so-called Kaplan classes in $\mathcal{H}(\mathbb{D})$, see below.

The starting point of the present investigation was a conjecture made by S.P. Robinson [6] (see also T. Sheil-Small [13, p. 300]). It looks rather technical and to state it we introduce the following notation, where $\mathcal{H}_0(\mathbb{D})$ is the set of functions $f \in \mathcal{H}(\mathbb{D})$ satisfying $f(0) = 1$.

Definition 1. For $\alpha, \beta > 0$ a function $f \in \mathcal{H}_0(\mathbb{D})$ is said to belong to the class[1] $\mathcal{T}(\alpha, \beta)^*$ if

$$f(z) * \frac{(1+xz)^{[\alpha]}(1+yz)^{\alpha-[\alpha]}}{(1-z)^{\beta}} \neq 0, \quad z \in \mathbb{D},\ x, y \in \overline{\mathbb{D}},$$

where $[\alpha]$ is the largest integer $\leqslant \alpha$.

Remark 1. A result of Sheil-Small [12, Lemma 2.1] shows that one can replace the condition $x, y \in \overline{\mathbb{D}}$ in Definition 1 by the weaker one $|x| = |y| = 1$ without changing the class $\mathcal{T}(\alpha, \beta)^*$.

Remark 2. One of the basic properties of the classes $\mathcal{T}(\alpha, \beta)^*$ is

$$\text{For } 1 \leqslant \alpha \leqslant \alpha',\ 1 \leqslant \beta \leqslant \beta' \text{ we have} \quad \mathcal{T}\left(\alpha', \beta'\right)^* \subset \mathcal{T}(\alpha, \beta)^* \tag{1}$$

(see f.i. [12, Thm. 1.3]).

Conjecture 1. *(See Robinson [6].) For $\alpha, \beta \geqslant 1$, $\gamma \geqslant \alpha + \beta$ and $n \in \mathbb{N}$ we have $C_n^{(\gamma-1)}(z) \in \mathcal{T}(\alpha, \beta)^*$.*

Remark 3. Actually, in [6] this conjecture was made only for $\gamma = \alpha + \beta$. But in view of (1) this implies the more general form given above.

Conjecture 1 is rather technically looking but has some interesting consequences, for instance in connection with the so-called Kaplan classes $\mathcal{K}(\alpha, \beta) \subset \mathcal{H}_0(\mathbb{D})$.

Definition 2. (Cf. Sheil-Small [12].) A function $f \in \mathcal{H}_0(\mathbb{D})$ belongs to $\mathcal{K}(\alpha, \beta)$ with $\alpha, \beta > 0$ if it has no zeros in $\mathbb{D}$ and satisfies the inequalities

$$\arg f\left(re^{i\theta_2}\right) - \arg f\left(re^{i\theta_1}\right) \geqslant -\alpha\pi - \frac{1}{2}(\alpha - \beta)(\theta_1 - \theta_2),$$

for all $0 < r < 1$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$.

The very strong relation between the classes $\mathcal{T}(\alpha, \beta)^*$ and $\mathcal{K}(\alpha, \beta)$ shows in the following result.

Theorem A. *(See [7,12].) Let $\alpha, \beta \geqslant 1$ and $f \in \mathcal{H}_0(\mathbb{D})$. Then $f \in \mathcal{T}(\alpha, \beta)^*$ if and only if $f * g \in \mathcal{K}(\alpha, \beta)$ holds for every $g \in \mathcal{K}(\alpha, \beta)$.*

One of our aims in this paper is to show that Conjecture 1 and the following Conjecture 2 are indeed equivalent.

Conjecture 2. *Let $\alpha, \beta \geqslant 1$, $\gamma \geqslant \alpha + \beta$. For $F \in \mathcal{H}_0(\mathbb{D})$ let*

$$\frac{F(zw)}{(1-w)^{\gamma}} := \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!} F_k^{(\gamma)}(z) w^k. \tag{2}$$

Then, $F \in \mathcal{K}(\alpha, \beta)$ if and only if $F_k^{(\gamma)}(z) \in \mathcal{K}(\alpha, \beta)$ holds for all $k \in \mathbb{N}$.

Remark 4. For the functions $F_k^{(\gamma)}$ in (2) we actually have

$$F_k^{(\gamma)}(z) = C_k^{(\gamma-1)}(z, F), \quad k \in \mathbb{N}.$$

It might be interesting to note that Conjecture 2, in turn, is equivalent to the following coefficient conjecture.

[1] This notation follows [8, p. 28]. In [6] the notation $T_0(\alpha, \beta)$ has been used instead, referring to the formal difference explained in Remark 1.

Conjecture 3. *Let*

$$\frac{(1-xw)^{[\alpha]}(1-yw)^{\alpha-[\alpha]}}{(1-w)^{\gamma}(1-uw)^{\beta}}=\sum_{k=0}^{\infty}B_k(\alpha,\beta,\gamma,x,y,u)w^k.$$

Then we have

$$B_k(\alpha,\beta,\gamma,x,y,u)\neq 0,\quad \alpha,\beta\geqslant 1,\ \gamma\geqslant\alpha+\beta,\ x,y,u\in\mathbb{D},\ k\in\mathbb{N}.$$

Remark 5. This reminds a bit of the so-called Brannan conjecture (and its unsolved parts), dealing with the estimation of the coefficients of the functions

$$\frac{(1+xz)^{\alpha}}{(1-z)^{\beta}}=\sum_{n=0}^{\infty}S_n(\alpha,\beta,x)z^n,\quad \alpha,\beta>0,\ x\in\overline{\mathbb{D}}$$

(see f.i. Brannan, Clunie and Kirwan [1] or Ruscheweyh and Salinas [10]).

The case $\alpha=1$, $\beta\geqslant 1$ of Conjecture 1 can be deduced from a more general result of Ruscheweyh [9]. This is our first theorem, for which we give a new and independent proof.

Theorem 1. *For $\gamma\geqslant\beta\geqslant 1$ we have $C_n^{(\gamma)}(z)\in\mathcal{T}(1,\beta)^*$, $n\in\mathbb{N}$.*

In all other cases the Robinson Conjecture seems to be very hard to deal with. Actually, very few members of the classes $\mathcal{T}(\alpha,\beta)^*$ with $\alpha>1$, $\beta>0$ are explicitly known. We quote T. Sheil-Small [13, p. 300] with the statement: "Finding even one non-trivial function in any of these classes is likely to yield significant results". The two main results in this paper are affirmative answers to the S.P. Robinson conjecture in special cases.

Theorem 2. *We have $C_n^{(2)}(z)\in\mathcal{T}(2,1)^*$, $n\in\mathbb{N}$.*

Theorem 3. *We have $C_n^{(3)}(z)\in\mathcal{T}(2,2)^*$, $n\in\mathbb{N}$.*

We mention in passing that [11, Thm. 3] together with the fact that, in the notation of [11],

$$F_{x,\beta}(z):=\frac{1}{2x+\beta}\left(\frac{(1+xz)^2}{(1-z)^{\beta}}-1\right)\in\mathcal{K}_{1-\beta/2},\quad |x|=1,\ 1\leqslant\beta\leqslant 2,$$

can be used to show that

$$C_n^{(\gamma)}\in\mathcal{T}(2,\beta)^*,\quad 1\leqslant\beta\leqslant 2,\ \gamma\geqslant\beta+2.$$

Since this does not seem to be sharp in any sense we omit the details.

1.2. Linear accessible functions and subordination

Theorem 3 is of special significance as we are going to explain in this section. In geometric function theory classes of univalent functions in $\mathcal{H}(\mathbb{D})$ whose image have special geometric properties are frequently studied. For instance, such a function is called 'linear accessible' if the complement of its image can be represented as the union of rays emanating from the boundary points of that image. Let us denote by $\mathcal{L}$ the set of those functions. Well-known members in $\mathcal{L}$ are the so-called close-to-convex functions (cf. Duren [2]), but $\mathcal{L}$ has indeed many more members and is not too well understood yet.

If $f,g\in\mathcal{H}(\mathbb{D})$ satisfy the conditions $f(0)=g(0)$ and $f=g\circ\varphi$, where $\varphi\in\mathcal{H}(\mathbb{D})$ fulfills $|\varphi(z)|\leqslant|z|$ for $z\in\mathbb{D}$, then we say f is subordinate to g, symbolically $f\prec g$.

We are interested in functions $f\in\mathcal{H}_0(\mathbb{D})$ which are range preserving over $\mathcal{L}$ in the sense that $(f*g)(\mathbb{D})\prec g(\mathbb{D})$ for any $g\in\mathcal{L}$. Denote the class of such functions by $\mathcal{U}(\mathcal{L})$.

The following characterization of $\mathcal{U}(\mathcal{L})$ is in [8, Thm. 2.23] (see the remark following the proof of that theorem, compare also [12]).

Theorem B. $\mathcal{U}(\mathcal{L})=\mathcal{T}(2,2)^*$.

The next theorem is therefore an immediate consequence of Theorem 3.

Theorem 4. *For $n \in \mathbb{N}$ we have $C_n^{(3)}(z) \in \mathcal{U}(\mathcal{L})$.*

Note that Theorem 4 does not imply the linear accessibility of $C_n^{(3)}(z, f)$ for $f \in \mathcal{L}$, in fact not even its univalence. Whether or not any of these properties hold in general is an open problem.

This paper is organized as follows. Sections 2 and 3 are devoted to the proofs of Theorems 1–3 and in Section 4 we show that Conjectures 1–3 are indeed equivalent.

2. Proof of Theorem 1

In view of (1) we can restrict our attention to the case $\gamma = \alpha + \beta$. Also, for $n = 0$ there is nothing to prove, hence we assume $n > 0$. We have to show that

$$C_n^{(\beta)} * \frac{1 + xz}{(1 - z)^\beta} \neq 0, \quad z \in \mathbb{D},\ |x| = 1.$$

A simple calculation shows that

$$C_n^{(\beta)} * \frac{z}{(1 - z)^\beta} = z\left(\frac{n}{\beta + n} C_{n-1}^{(\beta)}(z) * \frac{1}{(1 - z)^\beta}\right),$$

so that we have to show that

$$\left|\frac{(\beta + 1)_n}{n!} C_n^{(\beta)}(z) * \frac{1}{(1 - z)^\beta}\right| > |z| \left|\frac{(\beta + 1)_{n-1}}{(n - 1)!} C_{n-1}^{(\beta)}(z) * \frac{1}{(1 - z)^\beta}\right|, \tag{3}$$

for $|z| < 1$. First we prove

$$Q_n(z) := \frac{(\beta + 1)_n}{n!} C_n^{(\beta)}(z) * \frac{1}{(1 - z)^\beta} = \sum_{j=0}^{n} \frac{(\beta)_j}{j!} \frac{(\beta + 1)_{n-j}}{(n - j)!} z^j \neq 0, \quad z \in \mathbb{D}. \tag{4}$$

For this we invoke the following result of Lewis [4].

Lemma 1. *Let $0 \leqslant \lambda \leqslant \alpha + \beta$ and $\beta \geqslant \alpha > -\infty$. Then*

$$\sum_{j=0}^{n} \frac{(1 + \lambda)_{n-j}}{(n - j)!} \frac{(1 + \lambda)_j}{j!} \frac{P_j^{(\beta,\alpha)}(x)}{P_j^{(\beta,\alpha)}(1)} z^j \neq 0, \quad z \in \mathbb{D},\ n \in \mathbb{N},\ -1 \leqslant x \leqslant 1. \tag{5}$$

Here $P_n^{(\beta,\alpha)}(x)$ denotes a Jacobi polynomial in the standard notation. Now let $\lambda = \beta$, $\alpha = \beta - 1$, $x = -1$, and observe that

$$P_j^{(\beta,\alpha)}(1) = \frac{(1 + \beta)_j}{j!}, \qquad P_j^{(\beta,\alpha)}(-1) = (-1)^j \frac{(1 + \alpha)_j}{j!}.$$

Then (5) is exactly (4).

To complete the proof we use the following identity which is easily verified:

$$\frac{Q_n(z)}{Q_{n-1}(z)} = \frac{\beta + n}{n} + \frac{\beta z^n}{n} \frac{Q_{n-1}(1/z)}{Q_{n-1}(z)}.$$

Therefore, on $|z| = 1$,

$$\left|\frac{Q_n(z)}{Q_{n-1}(z)}\right| = \left|\frac{\beta + n}{n} + x\frac{\beta}{n}\right| \geqslant 1,$$

since

$$\left|\frac{Q_{n-1}(1/z)}{Q_{n-1}(z)}\right| = \left|\frac{\overline{Q_{n-1}(z)}}{Q_{n-1}(z)}\right| = |x| = 1.$$

The result follows by the minimum principle, applied to the function $Q_n(z)/Q_{n-1}(z)$ which is analytic and non-vanishing in $\mathbb{D}$.

3. Proof of Theorems 2 and 3

These proofs are rather technical and involved, mainly because of the need to show that certain trigonometric polynomials of high degree are non-negative. We refer to Remark 1 so that the condition to be checked is

$$C_n^{(\beta+1)}(z) * \frac{(1-xz)^2}{(1-z)^\beta} = \frac{n!}{(\beta+2)_n}\left(A_{0,n}^\beta(z) - 2xzA_{1,n}^\beta(z) + x^2z^2A_{2,n}^\beta(z)\right) \neq 0 \tag{6}$$

for $z \in \mathbb{D}$, $|x| = 1$, where

$$A_{j,n}^\beta(z) = \sum_{k=0}^{n-j} \frac{(\beta+2)_{n-j-k}}{(n-j-k)!} \frac{(\beta)_k}{k!} z^k.$$

It is clear that it will be sufficient to prove

$$F(z,y) := A_{0,n}^\beta(z) - 2yA_{1,n}^\beta(z) + y^2A_{2,n}^\beta(z) \neq 0, \quad z \in \overline{\mathbb{D}},\ y \in \mathbb{D}. \tag{7}$$

To do this we make use of two general results. The first one is due to Sheil-Small [12] and is as follows (simplified as to just meet our assumptions).

Lemma 2. *Let $F(z,y)$ be a polynomial in its variables such that $F(z,y) \neq 0$ for $|y| < 1$, $|z| = 1$ and $F(z,0) \neq 0$ for $z \in \overline{\mathbb{D}}$. Then, $F(z,y) \neq 0$ for $|y| < 1$, $z \in \overline{\mathbb{D}}$.*

Taking this into account we need to prove only

$$F(z,y) \neq 0, \quad |z| = 1,\ y \in \mathbb{D}, \quad \text{and} \tag{8}$$

$$F(z,0) \neq 0, \quad z \in \overline{\mathbb{D}}. \tag{9}$$

We now fix $n \in \mathbb{N}$ and $\beta \in \{1,2\}$ and write, for simplification,

$$a_j(z) := A_{j,n}^\beta(z), \quad j = 0,2; \qquad a_1(z) := -2A_{1,n}^\beta(z).$$

$F(z,y)$ is a polynomial of degree 2 in y. A simple reformulation of one of Cohn's rules (see [5, Thm. 11.5.3, Rule 2]), when applied to $f(y) := y^2F(z,1/y)$, shows that f has both its zeros in $|y| \leqslant 1$ (and therefore (8) is fulfilled) if the following inequalities are valid for $|z| = 1$:

$$D_1(z) := \left|a_0(z)\right|^2 - \left|a_2(z)\right|^2 > 0, \tag{10}$$

$$D_2(z) := \left(\left|a_0(z)\right|^2 - \left|a_2(z)\right|^2\right)^2 - \left|a_2(z)\overline{a_1(z)} - \overline{a_0(z)}a_1(z)\right|^2 \geqslant 0. \tag{11}$$

As for (9) we have to prove

$$a_0(z) \neq 0, \quad z \in \overline{\mathbb{D}}. \tag{12}$$

In the next two subsections we prove (10)–(12) for the cases $\beta = 1$ and $\beta = 2$ respectively.

3.1. The case $\beta = 1$

In what follows we make heavy use of computer algebra (Mathematica 5.2) to obtain the desired simplifications. We are not giving all details of the sometimes tricky ways to arrive at those simplified forms, but invite the reader to just check the final results by numerical or algebraic verification (note that all expressions involved are algebraic or trigonometric polynomials).

We have

$$a_j(z) = \sum_{k=0}^{n-j} \frac{(3)_{n-j-k}}{(n-j-k)!} z^k, \quad j = 0,2; \qquad a_1(z) = -2\sum_{k=0}^{n-1} \frac{(3)_{n-k-1}}{(n-k-1)!} z^k,$$

which leads to

$$D_1(1) = \frac{1}{3}(1+n)^3\left(3+2n+n^2\right) > 0,$$

so that for (10) we only need to look at $z \neq 1$.

Next we find

$$D_1(z)=\frac{n+1}{2(z-1)^3}\Big(\big((n+2)z+n\big)z^{n+2}-2\big(2n^2+4n+3\big)(z-1)z-\big(nz+(n+2)\big)z^{-n}\Big),$$

and then, writing $z=e^{i\varphi}$,

$$D_1\big(e^{i\varphi}\big)=-\frac{n+1}{8\sin^2\frac{\varphi}{2}}\left((n+2)\frac{\sin(2n+3)\frac{\varphi}{2}}{\sin\frac{\varphi}{2}}+n\frac{\sin(2n+1)\frac{\varphi}{2}}{\sin\frac{\varphi}{2}}-2\big(2n^2+4n+3\big)\right).$$

The expression in the large brackets on the right is obviously $\leqslant 0$, with equality only for $\varphi=0$. Hence $D_1(z)>0$ on $|z|=1$.

Similarly we obtain

$$D_2\big(e^{i\varphi}\big)=\frac{(n+1)^2}{32}\cdot\frac{\sin^2\frac{n+1}{2}\varphi}{\sin^6\frac{\varphi}{2}}T_n^{(1)}(\varphi),$$

where

$$\begin{aligned}T_n^{(1)}(\varphi)&=(n+2)^2\cos(n+2)\varphi-2n(n+2)\cos(n+1)\varphi+n^2\cos n\varphi\\&\quad-2\big(7n^2+14n+8\big)\cos\varphi+2\big(7n^2+14n+6\big).\end{aligned}$$

Lemma 3. *For $n\in\mathbb{N}$ and $\varphi\in(0,\pi]$ we have $T_n^{(1)}(\varphi)>0$.*

Proof. The first three terms of $T_n(\varphi)$ can be written as

$$\operatorname{Re}\big[e^{in\varphi}\big((n+2)e^{i\varphi}-n\big)^2\big],$$

which is greater or equal to

$$-\big|(n+2)e^{i\varphi}-n\big|^2=-\big((n+2)^2+n^2-2n(n+2)\cos\varphi\big),$$

so that our inequality is implied by

$$12n^2+24n+8>\big(12n^2+24n+16\big)\cos\varphi, \tag{13}$$

which is valid for those φ satisfying

$$\cos\varphi<1-\frac{2}{3(n+1)^2+1}.$$

This is easily seen to hold for $\varphi\in[\frac{\pi}{n+1},\pi]$. So it remains to prove the original inequality for $0<\varphi\leqslant\frac{\pi}{n+1}$. Note that $T_n^{(1)}(0)=0$. So it will be enough to show that the derivative of $T_n^{(1)}(\varphi)$ remains non-negative in $0<\varphi\leqslant\frac{\pi}{n+1}$. This is equivalent to

$$-(n+2)^3U_{n+1}(y)+2n(n+1)(n+2)U_n(y)-n^3U_{n-1}(y)\geqslant-14n^2-28n-16,$$

with $y=\cos\varphi$. Here U_n stands for the Chebyshev polynomial of the second kind and degree n. Using the relation

$$U_{n+1}(y)=2yU_n(y)-U_{n-1}(y)$$

we find the equivalent form

$$U_n(y)\big(2n(n+1)(n+2)-2y(n+2)^3\big)+U_{n-1}(y)\big((n+2)^3-n^3\big)\geqslant-14n^2-28n-16. \tag{14}$$

It is easily seen that

$$\sin(n\varphi)\geqslant\frac{n}{n+1}\sin\big((n+1)\varphi\big),\quad 0\leqslant\varphi\leqslant\frac{\pi}{n+1},$$

or, equivalently,

$$U_{n-1}\big(\cos(\varphi)\big)\geqslant\frac{n}{n+1}U_n\big(\cos(\varphi)\big),\quad 0\leqslant\varphi\leqslant\frac{\pi}{n+1},$$

with equality at $\varphi=0$. Inserting this into (14) we are left with

$$U_n(y)\left[2n(n+1)(n+2)-2y(n+2)^3+\frac{n}{n+1}\big((n+2)^3-n^3\big)\right]\geqslant-14n^2-28n-16. \tag{15}$$

Note that $U_n(y) \geqslant 0$ for the y in question. If the expression in the square brackets on the left is $\geqslant 0$ there nothing to prove. If it is < 0 then it will be smallest at $y = 1$ and $U_n(y)$ will be largest (namely $n + 1$) at the same point. Then, however, we have equality in (15). Lemma 3 is established. $\square$

A continuity argument completes the proof of $D_2(z) \geqslant 0$ on the whole of $|z| = 1$.

We now turn to the proof of (12). We have

$$a_0(z) = \frac{(n+1)(n+2)}{2} C_n^{(2)}(z),$$

and it is well known that the Cesàro means are convex-hull-preserving under convolution with functions in $\mathcal{H}(\mathbb{D})$. This implies $\operatorname{Re} C_n^{(2)}(z) > 1/2$ in $\mathbb{D}$. Therefore we can conclude that $a_0(z) \neq 0$ in $\overline{\mathbb{D}}$. The case $\beta = 1$ is complete.

3.2. The case $\beta = 2$

We follow the same pattern as in the previous case, but the details are even more involved. We have

$$a_j(z) = \sum_{k=0}^{n-j} \frac{(4)_{n-j-k}}{(n-j-k)!}(k+1)z^k, \quad j = 0, 2; \qquad a_1(z) = -2\sum_{k=0}^{n-1} \frac{(4)_{n-k-1}}{(n-k-1)!}(k+1)z^k,$$

and using this we find

$$D_1(1) = \frac{1}{720}(1+n)^2(2+n)^2(3+n)^2\big(20 + 18n + 6n^2 + n^3\big) > 0, \tag{16}$$

so that for (10) we only need to look at $z \neq 1$.

Using Mathematica and $z = e^{i\varphi}$ we get

$$D_1\big(e^{i\varphi}\big) = \frac{n+2}{768 \sin^8 \frac{\varphi}{2}} T_n^{(2)}(\varphi),$$

where

$$\begin{aligned} T_n^{(2)}(\varphi) = {} & (n+1)^2(n+3)\cos(n+4)\varphi - 2(n+1)\big(n^2+9n+12\big)\cos(n+3)\varphi \\ & + 18(n+1)(n+3)\cos(n+2)\varphi + 2(n+3)\big(n^2-n-8\big)\cos(n+1)\varphi \\ & - (n+1)(n+3)^2\cos n\varphi + 16(n+1)(n+3)\big(n^2+4n+5\big)\sin^4\frac{\varphi}{2} + 24. \end{aligned}$$

Lemma 4. *For $n \in \mathbb{N}$ and $\varphi \in (0, \pi]$ we have $T_n^{(2)}(\varphi) > 0$.*

Proof. For technical reasons we make the substitution $n \mapsto n - 2$. A rearrangement of the trigonometric expressions and division by a positive constant factor leaves us with the inequality

$$4\big(n^4 - 1\big)\sin^4\frac{\varphi}{2} + c_3(n,\varphi)n^3 + c_2(n,\varphi)n^2 + c_1(n,\varphi)n + c_0(n,\varphi) > 0 \tag{17}$$

where

$$\begin{aligned} c_3(n,\varphi) &= 2\sin\varphi\sin(n\varphi)\sin^2\frac{\varphi}{2}, \\ c_2(n,\varphi) &= \cos(n\varphi)(1-\cos\varphi)(\cos\varphi+5), \\ c_1(n,\varphi) &= -\sin\varphi\sin(n\varphi)(7-\cos\varphi), \\ c_0(n,\varphi) &= 6 - \cos(n\varphi)\big(6-(1-\cos\varphi)^2\big), \end{aligned}$$

which we have to establish for $n \geqslant 3$ and $\varphi \in (0, \pi]$.

In the case $n = 3$, after some rearrangements, the inequality (17) reduces to

$$64\left(\cos\varphi + \frac{5}{4}\right)(\cos\varphi - 1)^4 > 0$$

which is obviously true for all $\varphi \in (0, \pi]$.

In the case $n \geqslant 4$ of (17) we first note that the left-hand side vanishes at $\varphi = 0$. So, it will be sufficient to prove, that its derivative with respect to φ is non-negative for $\varphi \in [0, \pi]$. This derivative equals

$$2\left(n^2-1\right)\sin^2\frac{\varphi}{2}\left[\left(2\left(n^2+1\right)+\left(n^2-2\right)\cos(n\varphi)\right)\sin\varphi+n(\cos\varphi-4)\sin(n\varphi)\right],$$

so that we are left with the inequality

$$g(n,\varphi)=\left(2\left(n^2+1\right)+\left(n^2-2\right)\cos(n\varphi)\right)\sin\varphi-n(4-\cos\varphi)\sin(n\varphi)\geqslant 0 \tag{18}$$

to be established for $\varphi\in[0,\pi]$.

We divide the interval $[0,\pi]$ into 3 subintervals: $\mathrm{I}=[0,\frac{\pi}{n}]$, $\mathrm{II}=[\frac{\pi}{n},\frac{2\pi}{n}]$, $\mathrm{III}=[\frac{2\pi}{n},\pi]$, and study each of them separately.

I. Let $\varphi\in[0,\frac{\pi}{n}]$. Obviously the first term of $g(n,\varphi)$ is always non-negative in $[0,\pi]$. Furthermore, $\sin(n\varphi)\geqslant 0$ in the interval under consideration. Therefore, we can use the following approximations to estimate $g(n,\varphi)$:

$$\cos(n\varphi)\geqslant 1-\frac{n^2\varphi^2}{2}+\frac{n^4\varphi^4}{24}-\frac{n^6\varphi^6}{720},\qquad \sin\varphi\geqslant\varphi-\frac{\varphi^3}{6},$$
$$\sin(n\varphi)\leqslant n\varphi-\frac{n^3\varphi^3}{6}+\frac{n^5\varphi^5}{120},\qquad \cos\varphi\geqslant 1-\frac{\varphi^2}{2}.$$

Then we are left with the (essentially) biquadratic inequality

$$\frac{\varphi^5 n^2}{4320}\left(n^4\left(n^2-2\right)\varphi^4-6n^2\left(n^4+6n^2-10\right)\varphi^2+72\left(n^4+5n^2-10\right)\right)\geqslant 0,$$

to be verified in the subinterval I. This, however, is a simple matter of calculus.

II. If $\varphi\in[\frac{\pi}{n},\frac{2\pi}{n}]$, then $\sin(n\varphi)\leqslant 0$ and both terms on the left-hand side of (18) are non-negative. This means that (18), and consequently (17), are true in this sub-interval.

III. Let $\varphi\in[\frac{2\pi}{n},\pi]$. Since $|\cos(n\varphi)|\leqslant 1$ and $|\sin(n\varphi)|\leqslant 1$, it is sufficient to show, that

$$0<4\left(n^4-1\right)\sin^4\frac{\varphi}{2}-2\sin\varphi\sin^2\frac{\varphi}{2}n^3-2\sin^2\frac{\varphi}{2}(\cos\varphi+5)n^2$$
$$-\sin\varphi(7-\cos\varphi)n+6-(1+\sqrt{6}-\cos\varphi)(\sqrt{6}-1+\cos\varphi).$$

We substitute $t:=\tan\frac{\varphi}{2}$. Then, using $\sin^2\frac{\varphi}{2}=\frac{t^2}{1+t^2}$, $\sin\varphi=\frac{2t}{1+t^2}$ and $\cos\varphi=\frac{1-t^2}{1+t^2}$, we arrive at the equivalent inequality

$$h(n,t)=n\left(n^2-2\right)t^3-\left(n^2+4\right)t^2-3nt-3>0$$

to be proved for $t\in[\tan\frac{\pi}{n},+\infty)$. $h(n,t)$ is increasing with t in that interval, and it is easily checked that $h(n,\tan\frac{\pi}{n})>0$.

This completes the proof of Lemma 4 and of (10) for this case. □

Next we estimate $D_2(z)$ for $z=e^{i\varphi}$. Mathematica gives

$$D_2\left(e^{i\varphi}\right)=\frac{-(n+2)^2(n^2+4n+3)}{18432\sin^{12}\frac{\varphi}{2}}\left[(n+1)\sin(n+3)\frac{\varphi}{2}-(n+3)\sin(n+1)\frac{\varphi}{2}\right]^2 T_n^{(3)}(\varphi)$$

where

$$T_n^{(3)}(\varphi)=(n+1)(n+3)\cos(n+3)\varphi-2\left(n^2+4n+9\right)\cos(n+2)\varphi+(n+1)(n+3)\cos(n+1)\varphi$$
$$+2\left(5n^2+20n+21\right)\cos\varphi-10(n+1)(n+3).$$

Lemma 5. *For $n\in\mathbb{N}$ and $\varphi\in(0,\pi]$ we have $T_n^{(3)}(\varphi)<0$.*

Proof. We can rewrite the inequality in question as

$$(n+1)(n+3)\left[\cos(n+3)\varphi-2\cos(n+2)\varphi+\cos(n+1)\varphi\right]$$
$$<12\cos(n+2)\varphi-2\left(5n^2+20n+21\right)\cos\varphi+10(n+1)(n+3).$$

Using

$$\cos(n+3)\varphi-2\cos(n+2)\varphi+\cos(n+1)\varphi=-4\sin^2\frac{\varphi}{2}\cos(n+2)\varphi$$

we are left with

$$\left[3+(n+1)(n+3)\sin^2\frac{\varphi}{2}\right]\cos(n+2)\varphi+\left(5n^2+20n+21\right)\sin^2\frac{\varphi}{2}>3, \tag{19}$$

for $\varphi \in (0, \pi]$. Again we divide the interval $(0, \pi]$ into three subintervals

$$\mathrm{I} = \left(0, \frac{\pi}{2n+4}\right], \qquad \mathrm{II} = \left[\frac{\pi}{2n+4}, \frac{\pi}{n+2}\right], \qquad \mathrm{III} = \left[\frac{\pi}{n+2}, \pi\right]$$

and prove inequality (19) for each of them separately.

I. Let $\varphi \in (0, \frac{\pi}{2n+4}]$. As $\cos(n+2)\varphi$ is positive on this interval we can approximate the trigonometric functions, appearing in (19) as follows:

$$\cos(n+2)\varphi \geqslant 1 - \frac{(n+2)^2\varphi^2}{2} + \frac{(n+2)^4\varphi^4}{24} - \frac{(n+2)^6\varphi^6}{720},$$
$$\sin^2\frac{\varphi}{2} = \frac{1-\cos\varphi}{2} \geqslant \frac{\varphi^2}{4} - \frac{\varphi^4}{48}.$$

Substituting these into (19) we are left with

$$\left(1 - \frac{(n+2)^2\varphi^2}{2} + \frac{(n+2)^4\varphi^4}{24} - \frac{(n+2)^6\varphi^6}{720}\right)\left(3 + (n+1)(n+3)\left(\frac{\varphi^2}{4} - \frac{\varphi^4}{48}\right)\right)$$
$$+ \left(5n^2 + 20n + 21\right)\left(\frac{\varphi^2}{4} - \frac{\varphi^4}{48}\right) - 3 > 0.$$

An expansion of the left-hand side leads to the equivalent inequality

$$0 < \frac{(n+2)^2}{34560}\varphi^6\big[72\big(3n^4 + 24n^3 + 72n^2 + 96n + 43\big)$$
$$- 6(n+1)(n+2)^2(n+3)\big(2n^2 + 8n + 13\big)\varphi^2 + (n+1)(n+2)^4(n+3)\varphi^4\big].$$

That the expression in the brackets is positive for the range of φ in question is again just calculus, and we omit the details.

II. $\varphi \in [\frac{\pi}{2n+4}, \frac{\pi}{n+2}]$. Since $\cos(n+2)\varphi$ is negative on this interval we can estimate the trigonometric functions occurring in the inequality as follows:

$$\cos(n+2)\varphi \geqslant 1 - \frac{(n+2)^2\varphi^2}{2} + \frac{(n+2)^4\varphi^4}{24} - \frac{(n+2)^6\varphi^6}{720},$$

$\sin^2\frac{\varphi}{2}$ in its first occurrence by

$$\sin^2\frac{\varphi}{2} = \frac{1-\cos\varphi}{2} \leqslant \frac{\varphi^2}{4}$$

and in the second one by

$$\sin^2\frac{\varphi}{2} = \frac{1-\cos\varphi}{2} \geqslant \frac{\varphi^2}{4} - \frac{\varphi^4}{48}.$$

Substituting these estimates into (19) and doing some rearrangement it remains to prove that

$$\frac{\varphi^4}{48}\left[(n+1)(n+3) + \frac{(n+2)^4(3n^2+12n+7)}{10}\varphi^2 - \frac{(n+1)(n+2)^6(n+3)}{60}\varphi^4\right] > 0$$

for the given range of φ. Calculus shows that this is valid for $n \in \mathbb{N}$.

III. $\varphi \in [\frac{\pi}{n+2}, \pi]$. The expression in the brackets in (19) is positive and for all φ, n we have $\cos(n+2)\varphi \geqslant -1$. Therefore, it will be enough to prove

$$-\left(3 + (n+1)(n+3)\sin^2\frac{\varphi}{2}\right) + \left(5n^2 + 20n + 21\right)\sin^2\frac{\varphi}{2} - 3$$
$$= 2\left(\left(2n^2 + 8n + 9\right)\sin^2\frac{\varphi}{2} - 3\right) \geqslant 2\left(\left(2n^2 + 8n + 9\right)\sin^2\frac{\pi}{2n+4} - 3\right) > 0,$$

for $\varphi \in [\frac{\pi}{n+2}, \pi]$, which is easily established.

This completes the proof of Lemma 5 and, by continuity in $z = 1$, also of (11) for this case. $\square$

Finally we have to prove $a_0(z) \neq 0$, $z \in \overline{\mathbb{D}}$. This is obvious for $z=0$, and a simple calculation shows that

$$za_0(z) = \frac{(4)_{n+1}}{(n+1)!} z\big(C_{n+1}^{(3)}(z)\big)'.$$

It is known [3], that $C_n^{(3)}(z)$ is convex univalent in $\mathbb{D}$, for $n \in \mathbb{N}$. Hence $z(C_{n+1}^{(3)}(z))'$ is a starlike univalent polynomial in $\mathbb{D}$ and cannot have zeros in $\overline{\mathbb{D}} \setminus \{0\}$ (Koebe's one-quarter-theorem).

Theorems 2 and 3 have now been established.

4. The equivalence of Conjectures 1–3

4.1. Conjecture 1 ⇒ Conjecture 2

Let $\gamma \geqslant \alpha + \beta$. Assume Conjecture 1 is true. Then we have $C_n^{(\gamma-1)} \in \mathcal{T}(\alpha, \beta)^*$, $n \in \mathbb{N}$. Let $F \in \mathcal{K}(\alpha, \beta)$. By Theorem A this implies that $C_n^{(\gamma-1)}(z, F) \in \mathcal{K}(\alpha, \beta)$ for all $n \in \mathbb{N}$. The following formula is well known and easily verified:

$$\frac{1}{(1-w)^\gamma (1-zw)} = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!} C_k^{(\gamma-1)}(z) w^k. \tag{20}$$

It implies (by convolution) that

$$\frac{F(zw)}{(1-w)^\gamma} = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!} C_k^{(\gamma-1)}(z, F) w^k, \tag{21}$$

so that the $F_k^{(\gamma)}$ are indeed as in Remark 4 and Conjecture 2 is valid.

In view of (21) it also clear that the other direction of Conjecture 2 is valid, because the Cesàro means form approximate identities: $C_k^{(\gamma-1)}(z, F)$ converges locally uniformly in $\mathbb{D}$ to F, and $K(\alpha, \beta)$ is a compact set w.r.t. compact convergence.

4.2. Conjecture 2 ⇒ Conjecture 3

This part is immediate from the obvious fact that

$$F_{x,y,u} := \frac{(1-xz)^{[\alpha]}(1-yz)^{\alpha-[\alpha]}}{(1-uz)^\beta} \in \mathcal{K}(\alpha, \beta), \quad x, y, u \in \overline{\mathbb{D}},$$

and if Conjecture 2 (assumed to be correct) is applied to these functions.

4.3. Conjecture 3 ⇒ Conjecture 1

If Conjecture 3 holds, we find that

$$B_k(\alpha, \beta, \gamma, xz, yz, uz) = \frac{(\gamma)_k}{k!} C_k^{(\gamma-1)}(z, F_{x,y,u}) = \frac{(\gamma)_k}{k!} C_k^{(\gamma-1)}(z) * \frac{(1-xz)^{[\alpha]}(1-yz)^{\alpha-[\alpha]}}{(1-uz)^\beta} \neq 0,$$

for all $k \in \mathbb{N}$, $z \in \mathbb{D}$, $x, y, u \in \overline{\mathbb{D}}$. But this implies that $C_k^{(\gamma-1)}(z) \in \mathcal{T}(\alpha, \beta)^*$.

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